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THE PLANE CONTACT PROBLEM FOR AN ELASTIC LAYER FOR HIGH VIBRATION FREQUENCIES*

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The problem of stamp vibrations on the surface of an elastic strip located on a stiff base is examined. There is no friction in the contact domain or between the strip and the base. It is noted that the use of methods known earlier at high frequencies results in the need to solve linear algebraic systems of very high order. A method which enables the shortwave asymptotic form of the solution to be written in an explicit form convenient for qualitative and quantitative analyses is proposed.

1. We will assume that the time-dependence of all the functions occurring in the solution of the problem has the form $f(x, t) = \operatorname{Re}[f(x) e^{-i\omega t}]$ (ω is the angular frequency of the vibrations). Then the problem under investigation can be reduced to an integral equation in the unknown contact stress $p(x)$ referred to $\mu W/h$ /1/

$$\int_{-a}^{a} p(\xi) K(x-\xi)\, d\xi = 1, \quad |x| < a \tag{1.1}$$

$$K(x) = \frac{1}{2\pi} \int_{\Gamma} L(u) e^{-i\varkappa x u}\, du, \quad L(u) = L_1(u) - L_2(u)$$

$$L_1(u) = \sigma_1/\Delta(u), \; L_2(u) = \sigma_1 P_1(u)/\Delta(u)$$

$$P_1(u) = e^{-2\varkappa\sigma_1} + e^{-2\varkappa\sigma_2} - e^{-2\varkappa(\sigma_1+\sigma_2)}$$

$$\Delta(u) = 4u^2\sigma_1\sigma_2 G_1(u) F_2(u) - (2u^2-1)^2 G_2(u) F_1(u)$$

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$$G_k(u) = 1 - e^{-2\varkappa\sigma_k},\ F_k(u) = 1 + e^{-2\varkappa\sigma_k},\ k = 1,\ 2$$
$$\sigma_1 = \sqrt{u^2 - \beta^2},\quad \sigma_2 = \sqrt{u^2 - 1},\quad \beta^2 = \frac{1-2\nu}{2-2\nu} = \left(\frac{c_2}{c_1}\right)^2,$$
$$\varkappa^2 = \frac{\rho\omega^2 h^2}{\mu},\quad a = \frac{b}{h}$$

Here h is the layer thickness, b is the stamp half-width, μ and ν are the shear modulus and Poisson's ratio, and ρ is the density. To fix our ideas, the stamp base is assumed to be plane, and W is the amplitude of its vibrations.

Eq.(1.1) has been investigated earlier /1, 2/. It turns out that taking into account the zeros and poles of the symbol of the kernel $L(u)$ on the real axis is essential in constructing its solution. In the approaches used in /1, 2/ the problem is reduced to a linear algebraic system, where its order equals the number of positive zeros of the symbol α_k and β_k

$$\alpha_k = \sqrt{1 - (\pi k/\varkappa)^2}\ (k = 1, \ldots, n_\alpha),\quad \beta_k = \sqrt{\beta^2 - (\pi k/\varkappa)^2}\ (k = 1, \ldots, n_\beta) \tag{1.2}$$

As the vibration frequency increases, the parameter $\varkappa$ obviously increases and the number of real zeros of α_k and β_k increases with it. Therefore, systems of very high order must be solved at high frequencies in the approaches used in /1, 2/.

Another approach is proposed in this paper, based on an asymptotic analysis of (1.1) as $\varkappa \to \infty$. The second indepedent parameter a is here assumed to be fixed. The method is an extension of the method used in /3/ in which a simpler equation, for the problem of antiplane vibrations of a stamp, was investigated. For successful application to the problem being considered here, the method of /3/ requires cosiderable additional investigation.

2. We separate the function $K_1(x)$ with the symbol $L_1(u)$ possessing algebraic behaviour as $u \to \pm\infty$ and without zeros on the real axis, from the kernel $K(x)$. In such a decomposition the functions $L_1(u)$ and $L_2(u)$ that have branch points appear in place of the meromorphic function $L(u)$. Slits /1/ are made in a known manner to extract single-valued branches in the plane of the complex variable u.

We rewrite (1.1) in the form

$$\int_{-a}^{a} p(\xi) K_1(x-\xi)\, d\xi = 1 - \int_{-a}^{a} p(\xi) K_2(x-\xi)\, d\xi \tag{2.1}$$

We will formulate the problem of inverting the operator with the kernel $K_1(x)$ on the left side for (2.1). It is known /4, 5/ that as $\varkappa \to \infty$ the domain $(-a, a)$ decomposes into the main external domain and two small boundary-layer domains of length $\sim 1/\varkappa$, adjoining the ends of the interval. We will consider the global structure of the solution and its behaviour in the external domain. It follows from the results in /1/ that if the case of singular values of the parameter $\varkappa$, for which there are double zeros or poles on the real axis, is excluded, then the external solution can be obtained by extending the operator K_1 over the whole real axis. This is associated with the fact that the symbol $L_1(u)$ has no real zeros.

It is later assumed that the parameter $\varkappa$ can take any large values outside of small ε-neighbourhoods of the singular values mentioned. Then as $\varkappa \to \infty$ the operator K_1 is transformed into a convolution operator and its explicit inversion results in the equation

$$\int_{-a}^{a} p(\xi) Q(x-\xi)\, d\xi = l_0,\quad |x| < a \tag{2.2}$$
$$l_0 = \left.\frac{\Delta}{\sigma_1}\right|_{u=0} = -\frac{i}{\beta}(1 + e^{2i\varkappa\beta})(1 - e^{2i\varkappa})$$
$$Q(x) = \frac{1}{2\pi}\int_\Gamma G(u) e^{-i\varkappa ux}\, du,\quad G(u) = G_1(u) G_2(u)$$

Since the symbol $G(u)$ in (2.2) conserves all the zeros of the initial symbol $L(u)$, then the difficulty mentioned here, associated with the presence of these zeros, is not overcome in (2.2). Nevertheless, changing from (1.1) to (2.2) allows substantial simplification in obtaining the final solution. Moreover, as will be shown below, since (2.2) is an integral equation of the second kind with a continuous kernel, and its kernel can be expressed in elementary form $\varkappa \gg 1$ a direct method of solution can effectively be applied to this equation.

3. Taking account of the equality

$$\frac{1}{2\pi}\int_\Gamma e^{-i\varkappa ux}\, du = \frac{1}{\varkappa}\delta(x)$$

(2.2) can be reduced to the form

$$p(x) - \varkappa \int_{-a}^{a} p(\xi) Q_1(x-\xi)\, d\xi = l_0 \varkappa, \quad |x| < a \tag{3.1}$$

$$Q_1(x) = \frac{1}{2\pi} \int_{\Gamma} P_1(u) e^{-i\varkappa ux}\, du, \quad P_1(u) = 1 - G(u)$$

The kernel $Q_1(x)$ is obviously continuous (actually it is even infinitely differentiable).

It has been shown /3-5/ that an equation of the form (3.1) is equivalent to two equations in the new unknown functions $\varphi(x)$ and $v(x)$:

$$\varphi(x) - \varkappa \int_0^{\infty} \varphi(\xi) Q_1(x-\xi)\, d\xi = l_0\varkappa - \varkappa \int_0^{\infty} [\varphi(2a+\xi) - v] Q_1(x+\xi)\, d\xi, \quad x > 0 \tag{3.2}$$

$$v(x) - \varkappa \int_0^{\infty} v(\xi) Q_1(x-\xi)\, d\xi = l_0\varkappa, \quad |x| < \infty \tag{3.3}$$

if only

$$p(x) = \varphi(a+x) + \varphi(a-x) - v, \quad |x| < a \tag{3.4}$$

In problems where the symbol $G(u)$ has no zeros on the real axis, it is usually proved successfully /4, 5/ that $\varphi(x) \to v$ as $\varkappa \to \infty$. In this case the last integral in (3.2) turns out to be small and (3.2) and (3.3) become independent. Here (3.2) is transformed into a Wiener-Hopf equation and is solved by the factorization method, while (3.4) is a convolution equation and is solved in an elementary way by using a Fourier transformation.

We will prove the smallness of the integral in this problem by starting from the properties of the kernel $Q_1(x)$. Initially, we obtain an asymptotic expression for $Q_1(x)$ as $\varkappa \to \infty$. The main difficulty here is to estimate an integral of the form

$$J = \frac{1}{2\pi} \int_{\Gamma} e^{-i\varkappa ux} e^{-2\varkappa(\sigma_1+\sigma_2)}\, du$$

It can be shown that the main contribution to J is made by the neighbourhood of the stationary point u_* of the phase S:

$$S(u, x) = ux + 2\sqrt{1-u^2} + 2\sqrt{\beta^2 - u^2} \tag{3.5}$$

where $0 < u_* < \beta$. It can also be shown that S_u is a monotonically decreasing continuous function for any fixed $x > 0$, where $S_u > 0$ for $u = 0$ and $S_u < 0$ for $u = \beta - 0$. Therefore, the equation $S_u = 0$ determining the stationary point u_* always has a unique solution that is easily found numerically, for instance, by the method of half division. In summary, we obtain that

$$J \sim \frac{\exp(-i\pi/4)}{\sqrt{2\pi\varkappa}} \frac{\exp(i\varkappa S(u_*, x))}{\sqrt{|S_{uu}(u_*, x)|}}$$

as $\varkappa \to \infty$.

Using the asymptotic forms of the two other components in the kernel $Q_1(x)$ /3/, we have the estimate

$$Q_1 \sim \varkappa^{-1/2} [A_1(x) \exp(i\varkappa S(u_*, x)) + A_2(x) \exp(i\varkappa\beta \sqrt{4+x^2}) + A_3(x) \exp(i\varkappa \sqrt{4+x^2})] \tag{3.6}$$

where $A_1(x)$, $A_2(x)$ and $A_3(x)$ are smooth functions independent of $\varkappa$.

A further estimate of the integral on the right-hand side of (3.2) will be based on the use of integration by parts, well-known in such cases. Since the phase function $(4+x^2)^{1/2}$ has no stationary points for $x > 0$, such an integration by parts shows that the contribution from components corresponding to the functions A_2 and A_3 to the integral under consideration is of the order of $\varkappa^{-3/2}$. To explain the asymptotic behaviour of the component corresponding to the function A_1 we first see that there are no stationary points (in x) for the function $S(u_*, x)$ for $x > 0$. In fact, the equality

$$S_u(u, x) = 0 \tag{3.7}$$

determines the stationary point $u_* = u_*(x)$; consequently, $S(u_*, x) = S[u_*(x), x]$. It hence

follows that $dS/dx = S_u u_{*x} + S_x$, or taking (3.6) and (3.7) into account $dS/dx = S_x = u = u_*(x)$.
Therefore, $dS/dx = 0$ only for $u_* = 0$, and this is possible only for $x = 0$, as is proved directly. From this it follows that from the term with A_1 the contribution to the integral on the right-hand side of (3.2) is of the order $\varkappa^{-1/2}$. As $\varkappa \to \infty$ this integral can therefore be discarded.

4. Omitting the details of the solution of the Wiener-Hopf Eq.(3.2) and the convolution Eq.(3.3), we will write down the final result

$$p(x) = \frac{\varkappa}{\beta \operatorname{tg} \varkappa\beta} - \frac{i\pi l_0 \varkappa}{2G_-(0)} \left\{ \sum_{k=1}^{n_\alpha} \frac{kG_+(\alpha_k) H_k(x, 1)}{1 - \exp[-2\varkappa\sigma_1(\alpha_k)]} + \sum_{k=1}^{n_\beta} \frac{kG_+(\beta_k) H_k(x, \beta)}{1 - \exp[-2\varkappa\sigma_2(\beta_k)]} \right\} \tag{4.1}$$

$$H_k(x, \beta) = \{\exp[i\sqrt{(\varkappa\beta)^2 - (\pi k)^2}(a + x)] + \exp[i\sqrt{(\varkappa\beta)^2 - (\pi k)^2} \cdot (a - x)]\} \times [(\varkappa\beta)^2 - (\pi k)^2]^{-1}, \quad G(\alpha) = G_+(\alpha) G_-(\alpha)$$

(the function $G_+(\alpha)$ is analytic in the upper half-plane).
We will also present an exact form of the solution of the analogous antiplane problem since a misprint occurred in /3/

$$\tau(x) = \frac{\varkappa}{\operatorname{tg} \varkappa} - \frac{i\pi l_0 \varkappa}{2G_-(0)} \sum_{k=1}^{n} kG_+(\alpha_k) H_k(x, 1), \quad G(u) = 1 - \exp(-2\varkappa\sigma_2) \tag{4.2}$$

(the quantities σ_2, α_k are determined from (1.1) and (1.2)). The complex calculation of the factor $G_+(\alpha)$ of the function $G(\alpha)$ for $\varkappa \gg 1$ can be simplified. Namely, integrals of the form

$$G_+(\alpha) = \exp\left\{\frac{1}{2i\pi} \int_{i\infty}^{-i\infty} \frac{\ln[1 - \exp(-2\varkappa\sqrt{u^2 - \gamma^2})]}{u - \alpha} du\right\} \tag{4.3}$$

can be reduced to the form

$$G_+(\alpha) = \exp\left\{\sqrt{\frac{i\varkappa\gamma}{2}} \frac{\alpha}{\pi} \int_0^\infty \frac{\ln[1 - \exp(2i\varkappa\gamma - 2z)]}{\varkappa\alpha^2 + 2i\gamma z} \frac{dz}{\sqrt{z}}\right\} \tag{4.4}$$

($\gamma = 1$ or $\gamma = \beta$) by successive replacement of the variables $u = -it$, $t = [z(z + 2\gamma)]^{-1/2}$, $z = t/\varkappa$, deformation of the contour, extraction of the main contribution as $\varkappa \to \infty$ and subsequent replacement of the variable $t = iz$.
The integral in (4.4) is convenient for numerical realization since the integrand has no singularities for $z > 0$ and decreases exponentially as $z \to \infty$.
The amplitude of the contact stress of the antiplane problem is shown in Fig.1 for $\varkappa = 40$ (the lower part of the figure) and for $\varkappa = 150$ (the upper part). Fig.2 corresponds to the plane problem for $\varkappa = 80$. It was assumed everywhere that $a = 1, \nu = 0.3$. Solutions corresponding to the explicit asymptotic formulas (4.1) and (4.2) are represented by the solid lines, the results of the numerical solution of the simplified Eq.(3.1) and its analogous equation in /3/ by dashes, and results of the numerical solution of the initial integral equation by the dash-dot curves. All numerical solutions are obtained by the collocation method. The dashed curves in the upper part of Fig.1 differ from the solid lines only in the neighbourhood of the end of the interval.
The direct numerical solution of the initial equation is fraught with great difficulties. This is explained by the fact that as $\varkappa \gg 1$ its kernel is the sum of a delta-like and several strongly oscillating functions, which makes the process of calculation very unstable. The solution can be successfully constructed only in the antiplane case where the kernel of the initial equation /3/, unlike (1.1), has a fairly simple form. The instability of the calculation also requires a critical relation to the curves represented by the dash-dot lines; it is possible only to speak about their qualitative comparison with the curves obtained by stably realizable methods.
As the frequency of vibration $\varkappa$ increases, the contact stress diagram becomes more and more wavelike in nature. This phenomenon is explained by multiple rereflection of the rays from the bottom of the layer and differs from the analogous problem for a half-plane /5/, where the contact stress tends to a constant value as $\varkappa \to \infty$.

In conclusion, we note that the case when the strip is attached to the stiff base /1, 2/ can also be investigated by the method proposed in this paper.

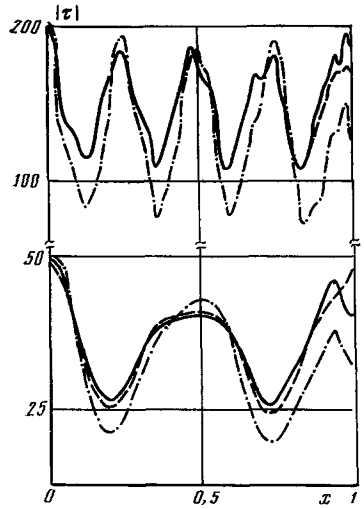

Fig.1

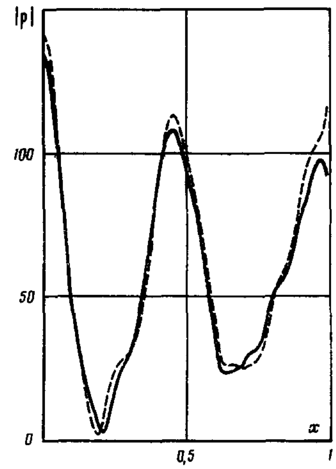

Fig.2

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